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DYNAMICS OF THE SPACE-TIME CORRELATIONS OF HYDRODYNAMIC FIELDS IN NON-STATIONARY GAS FLOWS*

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The Chapman-Enskog (CE) method of solving kinetic equations for the long-wave parts of the dual correlation functions of a non-equilibrium gas is developed. Closed systems of hydrodynamic equations are obtained, describing the dynamics of the space-time correlations of the large-scale hydrodynamic field fluctuations in non-stationary gas flows.

Two methods exist for constructing the basic hydrodynamic equations for the spatial correlations of fluctuations, based on kinetic theory. The first approach uses the kinetic stochastic equation for the fluctuations in the phase macrodensity of the one-particle states /1—3/. The CE method is used to obtain from it /4/ a closed system of equations for the fluctuations in the hydrodynamic fields with extraneous sources, whose correlations generalize the Landau-Lifshitz formula to embrace the range of non-equilibrium, but stable states of gas. The stochastic equations of hydrodynamics /4/ can, in principle, be used to obtain the equations describing the dynamics of the spatial correlations of the fluctuations of the hydrodynamic fields. The problem, arising in this case, of consecutive separation into terms of small and large scale spatial correlations is very complex and cumbersome in practice. The other approach based directly on the equations for dual correlation functions in a one-particle phase space /5, 6/ is found to be more effective. Thus the method of projection operators /5/ is used to construct a closed system of equations for the simultaneous spatial correlators of the hydrodynamic field fluctuations in inhomogeneous stationary states of a gas. Analogous equations were obtained in /6/ using the phenomenological approach.

The most important information concerning the character of the spatial statistical relations in laminar gas flows is contained in the relations describing the dependence of the inhomogeneous terms in the equations for the simultaneous spatial hydrodynamic correlators on the mean values of the thermodynamic forces representing the "heat sources" of the large-scale hydrodynamic correlations. The simplest linear terms of such a relation were computed in /5, 6/, and described incompletely the generation of spatial correlations in the non-isothermal flows of a compressible gas.

The purpose of this paper is to develop the CE method further in order to obtain the hydrodynamic asymptotic expression for the long-wave components of the dual correlation functions of the non-equilibrium gas, and to use them to construct a closed system of equations of dynamics for the space-time correlation functions of the fluctuations in the hydrodynamic fields. The two-point hydrodynamic equations obtained below for the space-time correlations are suitable, unlike the equations of /5, 6/, for studying thermal noise in unsteady compressible gas flows, and take into account the linear relations, as well as relations that are non-linear with respect to the gradients, describing the dependence of the "sources" of long-wave correlations on the mean values of the thermodynamic forces. The formal structure of the method developed here is universal in the sense that if the solution of the kinetic equation for the mean value of the corresponding phase macrodensity is known, then the solution of the kinetic equation for the long-wave part of the corresponding correlation function and the explicit form of the "source" terms in two-point hydrodynamic equations can be written down at once.

1. Initial kinetic equations and formulation of the problem.

Let us consider a simple, one-component, non-equilibrium gas. Let $f(t, x)$ and $\delta f(t, x)$ denote the mean value and fluctuations of the phase macrodensity of one-particle states $x = (r, v)$. Then the mean value $\Phi_\mu(t, r)$ and the fluctuations $\delta\Phi_\mu(t, r)$ $(\mu = 0, 1, \ldots, 4)$ of the hydrodynamic variables are given by the formulas

$$\Phi_\mu(t, r) = \int dv \Psi_\mu(t, x) f(t, x), \quad \delta\Phi_\mu(t, r) = \int dv \psi_\mu(t, x) \delta f(t, x) \tag{1.1}$$

$$\Psi_0 = m, \ \Psi_k = v_k/n, \ \Psi_4 = m(v-u)^2/(2n) \tag{1.2}$$

$$\psi_0 = m, \ \psi_k = (v_k - u_k)/n, \ \psi_4 = [m(v-u)^2/2 - e]/n \tag{1.3}$$

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$$(\mu = 0, 1, \ldots, 4; \quad k = 1, 2, 3; \quad n = \rho/m)$$

Here $\Phi_0 = \rho$, $\Phi_k = u_k$, $k = 1, 2, 3$, $\Phi_4 = e = 3k_BT/2$ are the mean values of the mass density, hydrodynamic velocity components and heat energy density, $\delta\Phi_0 = \delta\rho$, $\delta\Phi_k = \delta u_k$, $k = 1, 2, 3$, $\delta\Phi_4 = \delta e = 3k_BT/2$ are the fluctuations in the corresponding physical quantities, m is the mass of a single molecule, v is its velocity vector and k_B is Boltzmann's constant.

We know /1, 3/ that the one- and two-time correlations of the phase macrodensity can be written in the form

$$\langle\delta f(t, x_1)\, \delta f(t, x_2)\rangle = \delta(x_1 - x_2)\, f(t, x_1) + g(t, x_1, x_2) \tag{1.4}$$

$$\langle\delta f(t+\tau, x_1)\, \delta f(t, x_2)\rangle = F(t+\tau, x_1;\ t, x_2) + G(t+\tau, x_1;\ t, x_2) \tag{1.5}$$

The distribution function f and the long-wave part of the dual correlation function g are given by the equations

$$\mathrm{Kn}\left(\frac{\partial}{\partial t} + l_x\right) f(t, x) = J\,[f(t), f(t); x] \tag{1.6}$$

$$\mathrm{Kn}\left(\frac{\partial}{\partial t} + \sum_{i=1,2} l_{x_i}\right) g(t, x_1, x_2) = \sum_{i=1,2} J'\,[f(t); x_i]\, g(t, x_1, x_2) + \delta(r_1 - r_2)\, I\,[f(t), f(t); x_1, x_2], \quad l_x = v\cdot\nabla + E\cdot\frac{\partial}{\partial v} \tag{1.7}$$

Here $J\,[f, f; x]$ and $J'\,[f; x]$ denote the Boltzmann collision integral and its linearized operator. The functions F and G represent the two-time one-particle and two-time two-particle correlation functions. They satisfy the conditions

$$F(t, x_1; t, x_2) = \delta(x_1 - x_2)\, f(t, x_1),\ G(t, x_1; t, x_2) = g(t, x_1, x_2) \tag{1.8}$$

and the linearized Boltzmann equation, especially

$$\mathrm{Kn}\left(\frac{\partial}{\partial\tau} - l_{x_1}\right) G(t - \tau, x_1; t, x_2) = J'\,[f(t - \tau); x_1]\, G(t - \tau, x_1; t, x_2) \tag{1.9}$$

The Knudsen parameter Kn is introduced into the kinetic equations (1.6), (1.7), (1.9) to fix the order of magnitude of the separate terms at the hydrodynamic state of the evolution of the gaseous system.

From (1.1), (1.4) and (1.5) it follows that the correlation functions of the fluctuations in the hydrodynamic fields consist of two terms

$$\langle\delta\Phi_\mu(t, r_1)\, \delta\Phi_\nu(t, r_2)\rangle = a_{\mu\nu}(t, r_1, r_2) - b_{\mu\nu}(t, r_1, r_2) \tag{1.10}$$

$$\langle\delta\Phi_\mu(t - \tau, r_1)\, \delta\Phi_\nu(t, r_2)\rangle = \alpha_{\mu\nu}(t - \tau, r_1, t, r_2) - \beta_{\mu\nu}(t - \tau, r_1; t, r_2) \tag{1.11}$$

$(\mu, \nu = 0, 1, \ldots, 4)$ determined by the formulas

$$\begin{aligned} a_{\mu\nu}(t, r_1, r_2) &= \delta(r_1 - r_2)\int dv_1\,\psi_\mu(t, x_1)\,\psi_\nu(t, r_2, v_1)\, f(t, x_1) \\ b_{\mu\nv}(t, r_1, r_2) &= \int dv_1\, dv_2\,\psi_\mu(t, x_1)\,\psi_\nu(t, x_2)\, g(t, x_1, x_2) \\ \alpha_{\mu\nu}(t - \tau, r_1; t, r_2) &= \\ &\int dv_1\, dv_2\,\psi_\mu(t - \tau, x_1)\,\psi_\nu(t, x_2)\, F(t - \tau, x_1; t, x_2) \\ \beta_{\mu\nu}(t - \tau, r_1; t, r_2) &= \int dv_1\, dv_2\,\psi_\mu(t - \tau, x_1)\,\psi_\nu(t, x_2)\, G(t - \tau, x_1; t, x_2) \end{aligned} \tag{1.12}$$

and satisfy, by virtue of (1.8), the conditions

$$\alpha_{\mu\nu}(t, r_1; t, r_2) = a_{\mu\nu}(t, r_1, r_2), \quad \beta_{\mu\nu}(t, r_1; t, r_2) = b_{\mu\nu}(t, r_1, r_2) \tag{1.13}$$

The terms $b_{\mu\nu}$ and $\beta_{\mu\nu}$ appearing in expressions (1.10) and (1.11) for the hydrodynamic correlators, vanish when the state of thermodynamic equilibrium obtains. The terms $a_{\mu\nu}$ and $\alpha_{\mu\nu}$ determine the spatially δ-correlated part of the fluctuations describing the behaviour of the small-scale thermal fluctuations which do not vanish when the state of thermodynamic equilibrium obtains. They are studied in /7, 8/ for the case of a non-equilibrium gas.

The correlators $b_{\mu\nu}$ and $\beta_{\mu\nu}$ contain the most interesting information concerning the statistical properties of the non-equilibrium fluctuations in gas flows. They account for the large-scale fluctuations with a large correlation radius, existing only in non-equilibrium systems, and describe the long-wave statistical relations ensuring the presence of a fine, space-time statistical structure in the inhomogeneous flows. Namely, the part $b_{\mu\nu}$ of the correlation function (1.10) determines the influence of the large-scale fluctuations on the flow and the heat transfer in the gas /9/, due to the effects of molar transport. Moreover, it was shown in /10/ that the parts $b_{\mu\nu}$ and $\beta_{\mu\nu}$ of the hydrodynamic correlators account for the most characteristic features of the hydrodynamic fluctuations near the threshold of the convective stability of the flow. Namely, when the flow parameters approach their critical values corresponding to the loss of stability, the intensity of the spatial correlations and their radius both increases anomalously. We note that here the components $a_{\mu\nu}$ of the part of (1.10) have no singularities, and retain the order of magnitude of the quantity characterizing the thermal equilibrium noise level /8/, therefore they can be neglected in the neighbourhood of the critical non-equilibrium point.

We obtain the dynamic equations for the correlators $b_{\mu\nu}$ and $\beta_{\mu\nu}$ from the coupled systems of equations (1.6), (1.7) and (1.6), (1.9), and here we must consider the expanded sets of hydrodynamic variables Φ_μ, $b_{\mu\nu}$ $(\mu, \nu = 0, 1, \ldots, 4)$ for (1.6), (1.7) and Φ_μ, $\beta_{\mu\nu}$ $(\mu, \nu = 0, 1, \ldots, 4)$ for (1.6), (1.9). Then the classes of normal solutions of equations (1.6), (1.7), (1.9) describing the hydrodynamic stage of evolution of the gas will have the form

$$f(t, x) = f[\Phi(t); x],\ g(t, x_1, x_2) = g[\Phi(t), b(t); x_1, x_2] \qquad (1.14)$$
$$G(t + \tau, x_1; t, x_2) = G[\Phi(t + \tau), \beta(t + \tau, t); x_1, x_2]$$

2. Hydrodynamic asymptotic form of the long-wave parts of the paired correlation functions. We shall consider a class of normal solutions of (1.6), (1.7), asymptotic as $\mathrm{Kn} \to 0$, of the form

$$f = \sum_{n=0}^{\infty} (\mathrm{Kn})^n f^{(n)} [\Phi(t); x],\quad g = \sum_{n=0}^{\infty} (\mathrm{Kn})^n g^{(n)} [\Phi(t), b(t); x_1, x_2] \qquad (2.1)$$

In accordance with the scheme of the CE method we introduce the formal expansion of the time derivative into Eqs.(1.6), (1.7)

$$\frac{\partial}{\partial t} = \sum_{n=0}^{\infty} (\mathrm{Kn})^n \frac{\partial^{(n)}}{\partial t} \qquad (2.2)$$

and restrict the class of functions (2.1) by imposing the conditions of non-expandability of the hydrodynamic variables Φ and b

$$\int dv\, \Psi_\mu f^{(n)} = \delta_{n0} \Phi_\mu,\quad \int dv_1\, dv_2 \psi_\mu(x_1) \psi_\nu(x_2) g^{(n)} = \delta_{n0} b_{\mu\nu} \qquad (2.3)$$

The hydrodynamic equations for Φ and b obtained with help of such solutions of the kinetic equations (1.6), (1.7), have the following form in the N-th approximation:

$$\frac{\partial}{\partial t} \Phi_\mu = \sum_{n=0}^{N} (\mathrm{Kn})^n \frac{\partial^{(n)}}{\partial t} \Phi_\mu,\quad \frac{\partial}{\partial t} b_{\mu\nu} = \sum_{n=0}^{N} (\mathrm{Kn})^n \frac{\partial^{(n)}}{\partial t} b_{\mu\nu} \qquad (2.4)$$

Explicit expressions for $\partial^{(n)}\Phi/\partial t$ and $\partial^{(n)} b/\partial t$ are obtained from the conditions of solvability of the equations for $f^{(n+1)}$ and $g^{(n-1)}$ respectively.

We shall limit ourselves to constructing the first two approximations to the function g using known results /11/ of solving Boltzmann's equation (1.6) by the CE method. Let us introduce the notation $\partial^{(n)}\Phi_\mu(t, r)/\partial t = \Theta_\mu^{(n)}[\Phi(t); r]$. We shall write the Navier-Stokes equations in the form

$$\frac{\partial}{\partial t} \Phi_\mu(t, r) = \Lambda_\mu [\Phi(t); r] = \Theta_\mu^{(0)} [\Phi(t); r] - \Theta_\mu^{(1)} [\Phi(t); r] \qquad (2.5)$$

Substituting (2.1) and (2.2) into (1.7), we obtain the equations for the first three approximations to the function g

$$\sum_{i=1,2} J'[f^{(0)}; x_i] g^{(0)} = 0 \qquad (2.6)$$

$$\sum_{i=1,2} J'[f^{(0)}; x_i] g^{(1)} = \frac{d_2^{(0)}}{dt} g^{(0)} - \sum_{i=1,2} J'[f^{(1)}; x_i] g^{(0)} - \qquad (2.7)$$
$$\delta(r_1 - r_2) I[f^{(0)}, f^{(1)} + f^{(1)}, f^{(0)}]$$

$$\sum_{i=1,2} J'[f^{(0)}; x_i] g^{(2)} = \frac{d_2^{(0)}}{dt} g^{(1)} - \frac{\partial^{(1)}}{\partial t} g^{(0)} - \sum_{i=1,2} (J'[f^{(1)}; x_i] g^{(1)} + \qquad (2.8)$$
$$J'[f^{(2)}; x_i] g^{(0)}) - \delta(r_1 - r_2) I[f^{(0)}, f^{(2)} + f^{(2)}, f^{(0)} + f^{(1)}, f^{(1)}]$$
$$(d_2^{(0)}/dt = \partial^{(0)}/\partial t - l_{x_1} + l_{x_2})$$

The conditions of solvability of (2.7) and (2.8) (the orthogonality of their right sides with respect to the phase subspace stretched over the complete system (1.3) of the additive collision invariants) uniquely define the form of the first two terms on the right sides of Eqs.(2.4) for b. We note here that since the right-hand side of (1.7) is not orthogonal to the phase subspace stretched over the additive collision invariants, the resulting equation for b in the N-th approximation is found to be dependent on $f^{(N+1)}$ of the $(N+1)$-th approximation to the solution of Boltzmann's equation.

The following function represents a solution of (2.6) satisfying condition (2.3) at $n=0$:

$$g^{(0)}(t, x_1, x_2)=\sum_{\mu,\nu=0}^{4}\int dr_1'\, dr_2' b_{\mu\nu}(t, r_1', r_2')\frac{\delta f^{(0)}(t, x_1)}{\delta\Phi_\mu(t, r_1')}\times\frac{\delta f^{(0)}(t, x_2)}{\delta\Phi_\nu(t, r_2')}\equiv(b; \partial_\Phi f^{(0)}, \partial_\Phi f^{(0)}) \tag{2.9}$$

Here we denote by ∂_Φ the functional derivative $\delta/\delta\Phi_\mu(t, r)$, and the symbol $(\cdot;\cdot,\cdot)$ denotes the inner product of the functions within the brackets.

Before attempting to solve (2.7), we shall obtain an expression for $d_2^{(0)}g^{(0)}/dt$. It can be shown that

$$\begin{aligned}\frac{d_2^{(0)}}{dt}g^{(0)}=&\left(\left\{\frac{\partial^{(0)}}{\partial t}b-(\Theta^{(0)'}\cdot b+[\Theta^{(0)'}\cdot b]^*)\right\}; \partial_\Phi f^{(0)}, \partial_\Phi f^{(0)}\right)+\\&(b; \partial_\Phi J'[f^{(0)}]f^{(1)}, \partial_\Phi f^{(0)})-(b; \partial_\Phi f^{(0)}, \partial_\Phi J'[f^{(0)}]f^{(1)})\end{aligned} \tag{2.10}$$

Indeed, taking into account relation (2.9) we can write

$$\frac{d_2^{(0)}}{dt}g^{(0)}=\left(\frac{\partial^{(0)}}{\partial t}\Phi; \partial_\Phi\right)g^{(0)}+\left(\frac{\partial^{(0)}}{\partial t}b; \partial_\Phi f^{(0)}, \partial_\Phi f^{(0)}\right)+(l_{x_1}+l_{x_2})g^{(0)} \tag{2.11}$$

Taking into account the fact that the hydrodynamic variables Φ and b are independent, and also the equations

$$\frac{\partial^{(0)}}{\partial t}f^{(0)}=\left(\frac{\partial^{(0)}}{\partial t}\Phi; \partial_\Phi\right)f^{(0)}=(\Theta^{(0)}; \partial_\Phi)f^{(0)}$$

we can confirm the validity of the relation

$$\begin{aligned}&\left(b; \partial_\Phi\frac{\partial^{(0)}}{\partial t}f^{(0)}, \partial_\Phi f^{(0)}\right)+\left(b; \partial_\Phi f^{(0)}, \partial_\Phi\frac{\partial^{(0)}}{\partial t}f^{(0)}\right)=\\&(\Theta^{(0)'}\cdot b; \partial_\Phi f^{(0)}, \partial_\Phi f^{(0)})+([\Theta^{(0)'}\cdot b]^*; \partial_\Phi f^{(0)}, \partial_\Phi f^{(0)})+(\Theta^{(0)}; \partial_\Phi)g^{(0)}\end{aligned} \tag{2.12}$$

Here we have used tensor notation $\Theta^{(0)'}$ for the linearized Euler operator whose components are given by the formula

$$\Theta_{\mu,\alpha}^{(n)'}[\Phi; r]\varphi(r)=\int dr'\varphi(r')\,\delta\Theta_\mu^{(n)}[\Phi; r]/\delta\Phi_\alpha(r') \tag{2.13}$$

where $n=0$; $\Theta^{(0)'}\cdot b$ and $[\Theta^{(0)'}\cdot b]^*$ are tensors with components $\Theta_{\mu,\alpha}^{(0)'}[\Phi; r_1]\,b_{\alpha\nu}(t, r_1, r_2)$ and $\Theta_{\nu,\alpha}^{(0)'}[\Phi; r_2]\,b_{\mu\alpha}(t, r_1, r_2)$ where $\mu,\nu=0, 1, \dots, 4$ and the repeated index α denotes summation from 0 to 4.

From the equation for $f^{(1)}$ /11/ it follows that

$$\partial^{(0)}f^{(0)}/\partial t=-l_x f^{(0)}-J'[f^{(0)}]f^{(1)}$$

Let us use it in (2.12). The resulting expression will yield an expression for $(\partial^{(0)}\Phi/\partial t, \partial_\Phi)g^{(0)}$ which will then be substituted into (2.11). As a result we obtain (2.10).

We shall use (2.10) to calculate $\partial^{(0)}b/\partial t$. The condition for (2.7) to be solvable has the form

$$\begin{aligned}\int dv_1\, dv_2\psi_\mu(x_1)\psi_\nu(x_2)\frac{d_2^{(0)}}{dt}g^{(0)}=L_{\mu\nu}^{(1)}(r_1, r_2)\equiv\\\delta(r_1-r_2)\int dv_1\, dv_2\psi_\mu(x_1)\psi_\nu(x_2)I[f^{(0)}, f^{(1)}-f^{(1)}, f^{(0)}; x_1, x_2]\end{aligned} \tag{2.14}$$

The only non-zero contribution to it is given by the terms contained withint the first round brackets on the right-hand side of (2.10). Taking into account the fact that $\int dv\psi_\mu\partial_{\Phi_\nu}f^{(0)}=\delta_{\mu\nu}$, we obtain

$$\partial^{(0)}b_{\mu\nu}/\partial t=\Theta_{\mu,\alpha}^{(0)'}[\Phi; r_1]\,b_{\alpha\nu}+\Theta_{\nu,\alpha}^{(0)'}[\Phi; r_2]\,b_{\mu\alpha}+L_{\mu\nu}^{(1)} \tag{2.15}$$

Let us substitute (2.15) into (2.10) and use the equation

$$\partial_\Phi J'[f^{(0)}]f^{(1)}=J'[f^{(0)}]\partial_\Phi f^{(1)}+J'[f^{(1)}]\partial_\Phi f^{(0)}$$

Cancelling the like terms we obtain

$$\begin{aligned}d_2^{(0)}g^{(0)}/dt=&J'[f^{(0)}; x_1](b; \partial_\Phi f^{(1)}(x_1), \partial_\Phi f^{(0)}(x_2))+\\&J'[f^{(0)}; x_2](b; \partial_\Phi f^{(0)}(x_1), \partial_\Phi f^{(1)}(x_2))+\\&\sum_{i=1,2}J'[f^{(1)}; x_i]\,g^{(0)}+(L^{(1)}; \partial_\Phi f^{(0)}, \partial_\Phi f^{(0)})\end{aligned} \tag{2.16}$$

Taking into account (2.16), we transform (2.7) into

$$J'[f^{(0)};x_1]\{g^{(1)}-(b;\partial_\Phi f^{(1)},\partial_\Phi f^{(0)})\}+J'[f^{(0)};x_2]\times \tag{2.17}$$
$$\{g^{(1)}-(b;\partial_\Phi f^{(0)},\partial_\Phi f^{(1)})\}=M(t,x_1,x_2)\equiv$$
$$(L^{(1)};\partial_\Phi f^{(0)},\partial_\Phi f^{(0)})-\delta(r_1-r_2)I[f^{(0)},f^{(1)}+f^{(1)},f^{(0)};x_1,x_2]$$

Its solution, satisfying conditions (2.3) at $n=1$, has the form

$$g^{(1)}=(b,\partial_\Phi f^{(1)},\partial_\Phi f^{(0)})+(b;\partial_\Phi f^{(0)},\partial_\Phi f^{(1)})+g^{(1)'} \tag{2.18}$$

where $g^{(1)'}(t,\ x_1,\ x_2)$ is given by the equation

$$(J'[f^{(0)};\ x_1]+J'[f^{(0)};\ x_2])\,g^{(1)'}=M(t,\ x_1,\ x_2) \tag{2.19}$$

Let us now compute the second term on the right-hand side of Eq.(2.4) for *b*. We use the condition for (2.8) to be solvable

$$\int dv_1\,dv_2\psi_\mu(x_1)\,\psi_\nu(x_2)\left[\frac{d_2^{(0)}}{dt}g^{(1)}+\frac{\partial^{(1)}}{\partial t}g^{(0)}\right]=L^{(2)}_{\mu\nu}(r_1,r_2)\equiv \tag{2.20}$$
$$\delta(r_1-r_2)\int dv_1\,dv_2\psi_\mu(x_1)\,\psi_\nu(x_2)\,I[f^{(0)},f^{(2)}+f^{(2)},f^{(0)}+f^{(1)},f^{(1)}]$$

Taking into account (2.9), (2.18) and the equation given in /11/ for the second approximation $f^{(2)}$ to the solution of Boltzmann's equation by the CE method, and carrying out transformations analogous to those used in deriving (2.10), we can obtain

$$\frac{d_2^{(0)}}{dt}g^{(1)}+\frac{\partial^{(0)}}{\partial t}g^{(0)}=\left(\left\{\frac{\partial^{(1)}}{\partial t}b-(\Theta^{(1)'}\cdot b+[\Theta^{(1)'}\cdot b]^*)\right\};\partial_\Phi f^{(0)};\partial_\Phi f^{(0)}\right)+ \tag{2.21}$$
$$\sum_{i=1,2}l_{x_i}g^{(1)'}+(\Theta^{(0)},\partial_\Phi)\,g^{(1)'}+(L^{(1)};\partial_\Phi f^{(1)},\partial_\Phi f^{(0)})+$$
$$(L^{(1)};\partial_\Phi f^{(0)},\partial_\Phi f^{(1)})+\sum_{i=1,2}J'[f^{(1)};x_i]\{(b;\partial_\Phi f^{(1)},\partial_\Phi f^{(0)})+$$
$$(b;\partial_\Phi f^{(0)},\ \partial_\Phi f^{(1)})\}+\sum_{i=1,2}J'[f^{(2)};x_i]\,g^{(0)}$$

Let us substitute (2.21) into (2.20). When computing the integrals over the velocity space, we must take into account the relations

$$\int dv\psi_\mu(x)\,\partial_{\Phi_\nu}f^{(1)}=0,\qquad\int dv_1\,dv_2\psi_\mu(x_1)\,\psi_\nu(x_2)\,g^{(1)'}=0$$

which follow from (2.3), and use the fact that ψ_μ are the eigenfunctions of the operator $J'[f^{(0)}]$ corresponding to the fivefold degenerate zero eigenvalue. Here the non-zero contribution to (2.20) is made by the frist three terms of (2.21). As a result we obtain

$$\partial^{(1)}b_{\mu\nu}/\partial t=\Theta^{(1)'}_{\mu,\alpha}[\Phi;r_1]\,b_{\alpha\nu}+\Theta^{(1)'}_{\nu,\alpha}[\Phi;r_2]\,b_{\mu\alpha}+\Gamma^{(1)}_{\mu,\nu}(r_1,r_2)+ \tag{2.22}$$
$$\Gamma^{(1)}_{\nu,\mu}(r_1,r_2)+L^{(2)}_{\mu\nu}(r_1,r_2)$$

$$\Gamma^{(1)}_{\mu,\nu}(r_1,r_2)=-\int dv_1\,dv_2\psi_\mu(x_1)\,\psi_\nu(x_2)\,v_1\cdot\nabla_1 g^{(1)'} \tag{2.23}$$

The formulas (2.9), (2.18) together with (2.4) with $N=1$ for *b*, (2.15), (2.22) and the Navier-Stokes equations (2.5) together determine the required class of normal solutions of the kinetic equations (1.6), (1.7) in the first-order perturbation theory using the CE method. The system of hydrodynamic equations for the spatial correlators $b_{\mu\nu}(t,\ r_1,\ r_2)$, in this approximation, takes the form

$$\partial b_{\mu\nu}/\partial t-\Lambda'_{\mu,\alpha}[\Phi;r_1]\,b_{\alpha\nu}-\Lambda'_{\nu,\alpha}[\Phi;r_2]\,b_{\mu\alpha}=H_{\mu\nu}[\Phi;r_1,r_2] \tag{2.24}$$

$$H_{\mu\nu}(r_1,r_2)=\Gamma^{(1)}_{\mu,\nu}(r_1,r_2)+\Gamma^{(1)}_{\nu,\mu}(r_1,r_2)+L^{(1)}_{\mu\nu}(r_1,r_2)+L^{(2)}_{\mu\nu}(r_1,r_2) \tag{2.25}$$

It represents a system of "two-point" linearized inhomogeneous Navier-Stokes equations with the source terms $H_{\mu\nu}$ on the right-hand sides. In accordance with (2.25), (2.14), (2.20), (2.23) and (2.19), the terms depend on the special features of the small scale motions in an inhomogeneous gas, and vanish in the case of spatially homogeneous systems. In the latter case the trivial solutions of (2.24) are the only ones with any physical meaning. In the case of inhomogeneous systems the functions $H_{\mu\nu}$ represent heat sources of the hydrodynamic correlations. It should be noted that in the case of stationary states of an inhomogeneous gas Eqs. (2.24) differ from the corresponding equations of /5, 6/ in the structure of the inhomogeneous terms. In /5, 6/ the terms $\Gamma^{(1)}$ and $L^{(2)}$ appearing in (2.25) are not taken into account. They determine the sources of correlations representing special interest in the study of the fluctuations in the non-isothermal flows of a compressible gas.

The problem of constructing the normal solutions of (1.6), (1.9) of the form

$$G(t+\tau,x_1;t,x_2)=\sum_{n=0}^{\infty}(\mathrm{Kn})^n G^{(n)}[\Phi(t+\tau),\beta(t+\tau,t);x_1,x_2] \tag{2.26}$$

represents formally a special case of the problem discussed above. Its CE solution in the first-order perturbation theory can be written at once, taking the results obtained into account, as follows:

$$G=(\beta(t+\tau;t);\partial_{\Phi(t+\tau)}f^{(0)}(t+\tau,x_1),\partial_{\Phi(t)}f^{(0)}(t,x_2))+(\beta(t+\tau;t);\partial_{\Phi(t+\tau)}f^{(1)}(t+\tau,x_1),\partial_{\Phi(t)}f^{(0)}(t,x_2)) \tag{2.27}$$

At the same time, the equations of dynamics of the two-line hydrodynamic correlators β are identical with the linearized Navier-Stokes-Fourier equations

$$\begin{gathered}\partial\beta_{\mu\nu}/\partial\tau\,(t+\tau,\ r_1;\ t,\ r_2)-\Lambda'_{\mu,\alpha}[\Phi(t+\tau);\ r_1]\,\beta_{\alpha\nu}(t+\tau,r_1;\\ t,\ r_2)=0\end{gathered} \tag{2.28}$$

whose solutions should be considered, by virtue of (1.13), with the initial conditions determined by the solutions of (2.24), (2.5). Thus the Onsager principle determining the secular behaviour of the fluctuations in the macroscopic variables in an equilibrium system, can be generalized direclty to the domain of non-equilibrium, non-stationary, but stable hydrodynamic states of gaseous systems.

3. Computation of the sources of long-wave correlations in Eqs.(2.24). Let us consider the functional dependence of the inhomogeneous terms in Eqs.(2.24) on the mean values of the hydrodynamic fields. To do this, we first express them in terms of the integral brackets traditional in kinetic theory /11/.

Let us consider expression (2.23). Taking into account (2.3), we can reduce it to the form (repeated Latin indices denote summation from 1 to 3)

$$\begin{gathered}\Gamma^{(1)}_{\mu,\nu}(r_1,r_2)=\frac{\partial}{\partial r_{1p}}K_{p,\mu\nu}+\delta_{\mu l}K_{p,l\nu}\frac{1}{n(r_1)}\frac{\partial}{\partial r_{1p}}n(r_1)+\\ \delta_{\mu4}\left\{K_{p,l\nu}m\frac{\partial}{\partial r_{1p}}u_l(r_1)+K_{p,4\nu}\frac{1}{n(r_1)}\frac{\partial}{\partial r_{1p}}n(r_1)\right\}\end{gathered} \tag{3.1}$$

$$K_{p,\mu\nu}(r_1,r_2)=-\int dv_1\,dv_2\,\psi_\mu(x_1)\,\psi_\nu(x_2)\,c_{1p}g^{(1)\prime},\quad c_{1p}=v_{1p}-u_{1p}(r_1) \tag{3.2}$$

The explicit form of the function $g^{(1)\prime}$ is not required in computing the integrals (3.2). We shall show this as follows. Consider its equation (2.19). Multiplying both sides by $\psi_\nu(x_2)$ and integrating over v_2, we obtain

$$\begin{gathered}\int dv_2\psi_\nu(x_2)\,g^{(1)\prime}=\int dv_2\psi_\nu(x_2)\,(J'[f^{(0)};x_1])^{-1}\times\\ \{(L^{(1)};\partial_\Phi f^{(0)}(x_1),\partial_\Phi f^{(0)}(x_2))-\\ \delta(r_1-r_2)\,I[f^{(0)},f^{(1)}+f^{(1)},f^{(0)};x_1,x_2]\}\end{gathered} \tag{3.3}$$

Let us multiply both sides of this equation by $[c_{1p}\psi_\mu(x_1)-c_1{}^2\delta_{\mu p}/(3n)-k_BTc_{1p}\delta_{\mu4}/n]$ and integrate it over v_1. The second and third term on the left-hand side both vanish, and the first term yields (3.2). Taking into account on the right-hand side of (3.3) the selfconjugate nature of the operator $(J'[f^{(0)}])^{-1}$, we use the relation

$$\begin{gathered}(J'[f^{(0)}])^{-1}\left\{c_p\psi_\mu-\frac{c^2}{3n}\delta_{\mu p}-\frac{k_BT}{n}c_p\delta_{\mu4}\right\}=\\ -\frac{k_BT}{n^2}\left\{\frac{1}{m}B_{pl}\delta_{\mu l}+A_p\delta_{\mu4}\right\}\\ \left(A_p=-n\,(J'[f^{(0)}])^{-1}c_p\left(\frac{mc^2}{2k_BT}-\frac{5}{2}\right),\right.\\ \left.B_{pl}=-\frac{\rho}{k_BT}(J'[f^{(0)}])^{-1}\left(c_pc_l-\frac{1}{3}c^2\delta_{pl}\right)\right)\end{gathered}$$

remembering also the condition of non-expandability of the hydrodynamic variables. It can be shown here that all terms containing $L^{(1)}$ vanish. As a result we obtain

$$\begin{gathered}K_{p,\mu\nu}(r_1,r_2)=-\delta(r_1-r_2)\frac{k_BT}{n^2}\int dv_1\,dv_2\psi_\nu(x_2)\times\\ \left\{\frac{1}{m}B_{pl}(x_1)\,\delta_{\mu l}+A_p(x_1)\,\delta_{\mu4}\right\}I[f^{(0)},f^{(1)}+f^{(1)},f^{(0)};x_1,x_2]\end{gathered}$$

Introducing the definition of the modified integral brackets for the three arbitrary phase functions $R(v)$, $H(v)$, $G(v)$

$$\begin{gathered}[R;H,G]^*=-\frac{1}{n^2}\int dv_1R(v_1)\,J[f^{(0)}H,f^{(0)}G+f^{(0)}G,f^{(0)}H]=\\ -\frac{1}{n^2}\int dv_1\,dv_2H(v_1)\,G(v_2)\,I[f^{(0)},f^{(0)}R+f^{(0)}R,f^{(0)};x_1,x_2]\end{gathered} \tag{3.4}$$

we can write this expression in the form

$$K_{p,\,\mu\nu}(r_1,r_2)=\delta(r_1-r_2)\,k_BT\left\{\frac{1}{m}\delta_{\mu l}\,[h^{(1)};B_{pl},\psi_\nu]^*+\right.$$
$$\left.\delta_{\mu 4}[h^{(1)};A_p,\psi_\nu]^*\right\},\quad h^{(1)}=f^{(1)}/f^{(0)} \tag{3.5}$$

We transform expressions (2.14) and (2.20) for $L^{(1)}$ and $L^{(2)}$, using the formula

$$\int dv_1\,dv_2\psi_\mu(x_1)\,\psi_\nu(x_2)\,I\,[f,f;x_1,x_2]=-\int dv_1\psi_\mu(x_1)\,\psi_\nu(x_1)\,J\,[f,f;x_1]$$

which follows from formula (1.3) of /4/, and taking into account the definition of the standard integral bracket /11/

$$[R;H]=-\frac{1}{n^2}\int dvH(v)\,J'\,[f^{(0)};x]\,f^{(0)}R(v) \tag{3.6}$$

As a result we obtain

$$L^{(1)}_{\mu\nu}(r_1,r_2)=\delta(r_1-r_2)\,n^2\,[h^{(1)};\psi_\mu\psi_\nu] \tag{3.7}$$
$$L^{(2)}_{\mu\nu}(r_1,r_2)=\delta(r_1-r_2)\,n^2\left\{[h^{(2)};\psi_\mu\psi_\nu]+\frac{1}{2}[\psi_\mu\psi_\nu;h^{(1)},h^{(1)}]^*\right\}$$
$$h^{(2)}=f^{(2)}/f^{(0)}$$

Thus, further calculations can be carried using the methods employed in computing the integral brackets /11/ developed thoroughly in the kinetic theory of gases. Omitting the details, we shall give the results of these calculations, restricting ourselves for simplicity to the first terms of the expansions of the function defining $h^{(1)}$ and $h^{(2)}$ in terms of Sonin polynomials (a Maxwell gas). Some of the components of the tensors (3.5), (3.7) are equal to zero in this approximation. The non-zero components have the form

$$K_{p,\,l0}(r_1,r_2)=\delta(r_1-r_2)\frac{1}{n}P^{(1)}_{pl},\quad K_{p,40}(r_1,r_2)=\delta(r_1-r_2)\frac{m}{n}q^{(1)}_p \tag{3.8}$$
$$K_{r,\,lp}(r_1,r_2)=\delta(r_1-r_2)\frac{4}{15}\frac{1}{n\rho}E^{ps}_{rl}q^{(1)}_s$$

$$L^{(1)}_{pl}(r_1,r_2)=\delta(r_1-r_2)\frac{p^{(0)}}{\rho n\eta}P^{(1)}_{pl}, \tag{3.9}$$
$$L^{(1)}_{p4}(r_1,r_2)=\delta(r_1-r_2)\frac{5}{2}\frac{k_Bp^{(0)}}{\rho n\lambda}q^{(1)}_p$$

$$L^{(2)}_{pl}(r_1,r_2)=\delta(r_1-r_2)\frac{p^{(0)}}{\rho n\eta}P^{(2)}_{pl}, \tag{3.10}$$
$$L^{(2)}_{p4}(r_1,r_2)=\delta(r_1-r_2)\frac{5}{2}\frac{k_Bp^{(0)}}{\rho n\lambda}q^{(2)}_p$$
$$L^{(2)}_{44}(r_1,r_2)=\delta(r_1-r_2)\frac{2}{n^2}\left\{k_BT\,(P^{(1)}_{pl}\nabla_pu_l+\nabla_pq^{(1)}_p)-\right.$$
$$\left.\frac{1}{n}q^{(1)}_l\nabla_lp^{(0)}+mq^{(1)}_pE_p+\frac{7}{2}k_Bq^{(1)}_p\nabla_pT\right\}$$
$$p^{(0)}=nk_BT,\quad P^{(1)}_{rl}=-2\eta E^{rl}_{tp}\nabla_tu_p,\quad q^{(1)}_p=-\lambda\nabla_pT$$
$$E^{tp}_{rl}={}^1/_2(\delta_{pr}\delta_{tl}+\delta_{pl}\delta_{tr})-{}^1/_3\delta_{tp}\delta_{rl}$$

Here η,λ are the coefficient of viscosity and thermal conductivity $P_{rl}{}^{(2)}$ and $q_p{}^{(2)}$ are the Barnett complements to the tangential stress tensor and thermal flux /11/, δ_{pl} is the Kronecker delta.

From formulas (2.5), (3.1) and (3.8), (3.9), (3.10) it follows that the intensity of the thermal sources of long-wave correlations in the gas increase as its non-equilibrium character increases. The terms (3.9) in (2.25) are of zero order of magnitude relative to the Knudsen number, (3.1), (3.8) and (3.10) are of the first order. Formulas (3.10) describe the contribution of the terms of the Barnett approximation for the mean values of thermodynamic fluxes, to the correlation sources. In the case of flows described satisfactorily by the Navier-Stokes-Fourier equations, these terms can be neglected in (2.25). Here we have

$$H_{\mu\nu}(r_1,r_2)=\delta_{\mu p}\delta_{\nu l}\delta(r_1-r_2)\frac{p^{(0)}}{\rho n\eta}P^{(1)}_{pl}+[\delta_{\mu 0}\delta_{\nu 4}+$$
$$\delta_{\mu 4}\delta_{\nu 0}]\,\delta(r_1-r_2)\frac{m}{n}P^{(1)}_{pl}\nabla_pu_l+[\delta_{\mu 4}\delta_{\nu r}+\delta_{\mu r}\delta_{\nu 4}]\,\delta(r_1-r_2)\times$$
$$\frac{1}{n}\left[\frac{5}{2}\frac{k_Bp^{(0)}}{\rho\lambda}q^{(1)}_r+\frac{4}{15}\frac{1}{n}q^{(1)}_lE^{rl}_{tp}\nabla_tu_p\right]+$$
$$\left[\delta_{\mu 0}\delta_{\nu 4}\frac{1}{n(r_2)}\nabla_{2p}+\delta_{\mu 4}\delta_{\nu 0}\frac{1}{n(r_1)}\nabla_{1p}\right]\delta(r_1-r_2)\,mq^{(1)}_p+$$
$$\left[\delta_{\mu 0}\delta_{\nu p}\frac{1}{n(r_2)}\nabla_{2l}+\delta_{\mu p}\delta_{\nu 0}\frac{1}{n(r_1)}\nabla_{1l}\right]\delta(r_1-r_2)\,P^{(1)}_{pl}+$$
$$\delta_{\mu r}\delta_{\nu l}\left[\frac{1}{n(r_1)}\nabla_{1t}+\frac{1}{n(r_2)}\nabla_{2t}\right]\delta(r_1-r_2)\frac{4}{15}\frac{1}{\rho}E^{rl}_{tp}q^{(1)}_p \tag{3.11}$$

Equations (2.24) with inhomogeneous terms (3.11) do not contain any non-equilibrium parameters which characterize only the gaseous systems. Therefore they can be used to study non-equilibrium, large-scale fluctuations in fluid flows.

REFERENCES

1. KLIMONTOVICH YU.L., Kinetic Theory of a Non-Ideal Gas and Non-Ideal Plasma. Moscow, Nauka, 1975.
2. LIFSHITZ E.M. and PITAEVSKII L.P., Physical Kinetics. Moscow, Nauka, 1979.
3. GRECHANNYI O.A., On the theory of small fluctuations in a non-equilibrium gas. In book: Teplofizika i teplotekhnika, Ed.36, Kiev, Naukova Dumka, 1979.
4. GRECHANNYI O.A. and TOKARCHUK V.V., On the kinetic theory of thermal hydrodynamic fluctuations in an inhomogeneous gas. PMM, 45, 1, 1981.
5. KIRKPATRICK T.R., COHEN E.G.D. and DORFMAN J.R., Fluctations in a non-equilibrium steady states: Basic equations. Phys. Rev. A, 26, 2, 1982.
6. RONIS D., PROCACCIA I. and MACHTA J., Statistical mechanics of stationary states. VI. Hydrodynamics fluctuations theroy far from equilibrium. Phys. Rev. A, 22, 2, 1980.
7. FISHER I.Z. and LESNIKOV V.P., Non-equilibrium thermal hydrodynamic fluctuations. In book: Fizika zhidkogo sostoyaniya (Physics of the Liquid State), Ed.4, Kiev, Vishcha shkola, 1976.
8. LESNIKOV V.P. and FISHER I.Z., Thermal fluctuations and molecular dispersion of light in a liquid near the state of convective instability. ZhETF, 67, 4, 1974.
9. KHON'KIN A.D., On the influence of non-equilibrium fluctuations on the onset of turbulence. In book: Turbulentnye techeniya. (Turbulent Flow), Moscow, Nauka, 1974.
10. GRECHANNYI O.A. and TOKARCHUK V.V., Thermal hydrodynamic fluctuations near the state of convective Rayleigh-Benard instability. In book: Konvektivnyi teploperenos. Kiev, Naukova Dumka, 1982.
11. FERZIGER J.H. and KAPER H.G., Mathematical Theory of Transport Processes in Gases. London, North-Holland Publ. Co. 1972.

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ASYMPTOTIC FORM OF SMALL DENSITY DIFFERENCES IN THE PROBLEM OF COHERENT PHASE TRANSFORMATIONS*

M.A. GRINFEL'D

Equations describing (in the lower approximation) the equilibrium configurations under heterogeneous, coherent phase transformations in an elastic, one-component medium, are derived for the asymptotic case of small density differences. Both phases are assumed to be isotropic by virtue of the multiplicity and certain computational simplifications. It is shown that, to a first approximation, the equilibrium temperature of the non-hydrostatic, two-phase configuration is identical with the temperature of phase equilibrium of the hydrostatically stressed phases in some reference configuration. In a higher approximation the system of equations of equilibrium obtained is identical with the equations of the classical linear theory of elasticity, although, on the whole, the problem remains essentially non-linear, since it contains an unknown boundary and certain boundary conditions on it, quadratic with respect to the displacement. The conditions obtained are further used to find the solutions of certain boundary value problems.

The conditions of equilibrium obtained in /1, 2/ under coherent phase transformations with slippage, represent special boundary value problems for the equations of the non-linear theory of elasticity, with unknown boundaries. The presence of unknown boundaries of contact between the different phases aggravates the difficulties of the already complicated problem of describing the equilibrium configurations of non-linearly elastic materials (e.g. in the simplest problem of this type for a liquid system where the problem reduces to that of determining the equilibrium values of the pressures, temperture and phase masses, the equilibrium

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